Chaotic character of two-soliton collisions in the weakly perturbed nonlinear Schrödinger equation

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We analyze the exact two-soliton solution to the unperturbed nonlinear Schrödinger equation and predict that in a *weakly* perturbed system (i) soliton collisions can be *strongly* inelastic, (ii) inelastic collisions are of almost nonradiating type, (iii) results of a collision are extremely sensitive to the relative phase of solitons, and (iv) the effect is independent on the particular type of perturbation. In the numerical study we consider two different types of perturbation and confirm the predictions. We also show that this effect is a reason for chaotic soliton scattering. For applications, where the inelasticity of collision, induced by a weak perturbation, is undesirable, we propose a method of compensating it by perturbation of another type.

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I. INTRODUCTION

Influence of perturbations of various types on interaction of solitons in different integrable nonlinear equations has been studied thoroughly during last decades [1,2]. Different effects such as inelasticity of soliton collisions, mutual trapping, collapse of solitons etc. have been reported as common sequences of violated integrability [1–11]. It has been also reported that perturbation can be responsible for chaotic soliton scattering [11–17]. In the vast majority of studies, the inelastic soliton collisions and other effects of perturbation are related to the existence of solitons' internal modes (see, e.g., Ref. [10]). The internal modes can play an important role only if the perturbation parameter is not too small. As a consequence, the reported effects are usually accompanied by significant radiation of energy and the case of a weak perturbation is usually considered as the case of a small importance. However, a nontrivial effect of perturbation that is almost radiationless energy exchange between colliding solitons has been reported for weakly discrete sine-Gordon equation (SGE) [9,15] and nonlinear Schrödinger equation (NLSE) with small quintic perturbation [8]. In both cases, the effect was observed in *three*-soliton collisions. Recently, the strongly inelastic radiationless collision of *two* solitons was observed numerically in NLSE for weak discreteness [16,18]. We suppose that strongly inelastic collision is possible when the number of soliton components is greater than the number of quantities, conserved (with a high accuracy) for the perturbed equation. In SGE, the inelastic collision can be observed when at least three one-component solitons meet at one point, because there are two conserved quatities, momentum and energy [9,15]. NLSE soliton is a two-component one and there are three conserved quantities, norm, momentum, and energy. Thus, the conservation laws do not forbid a strongly inelastic collision between two NLSE solitons [16]. This is physically important because two-soliton collisions in NLSE are more probable compared to three-soliton collisions. The same concerns SGE, where three- and four-soliton collisions are easily observed in kink-breather and breather-breather collisions [9,15].

The aim of this paper is to show that in perturbed NLSE even two-soliton collisions can be strongly inelastic with a negligible amount of radiation and that the effect is essentially determined by the internal parameters of solitons but not by the particular type of perturbation. We also show that this effect is a reason for chaotic soliton scattering.

The perturbation parameter is taken to be extremely weak so that internal and radiative modes cannot play any significant role. Therefore, the traditional mechanisms of inelastic scattering (Refs. [8,11,12,14]) should be excluded from consideration.

On this purpose we integrate numerically the set of discrete NLSE with small discreteness parameter $\Delta\tau$ and small quintic term ($\epsilon \ll 1$):

$$i\frac{d\psi_n}{d\xi}+\frac{1}{2\Delta\tau^2}(\psi_{n-1}-2\psi_n+\psi_{n+1})+|\psi_n|^2\psi_n=\epsilon|\psi_n|^4\psi_n . \tag{1}$$

Besides numerous applications of the discrete NLSE in different fields of physics, e.g., nonlinear optics, dynamics of biomolecules, self-trapping phenomena etc., it is also used to analyze different types of localized modes in the discrete version of SGE [2].

Assuming that $\Delta\tau$ is small, one can transform Eq. (1) into following continuum perturbed NLSE

$$i\psi_\xi+\frac{1}{2}\psi_{\tau\tau}+|\psi|^2\psi=\epsilon|\psi|^4\psi-\frac{\Delta\tau^2}{24}\psi_{\tau\tau\tau\tau}. \tag{2}$$

The first term in the right-hand side of Eq. (2) is the quintic perturbation, important in nonlinear optics [1,3,8]. The second term is the leading term accounting for the discreteness of Eq. (1).

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In the absence of perturbation ($\epsilon=0, \Delta\tau\to 0$), NLSE is known to be exactly integrable, and it supports propagation of envelope solitons, which recover their properties after collision with each other, i.e., collide elastically. The integrable unperturbed NLSE has an infinite number of conservation laws. The conservation laws, important for analysis of soliton collisions, are the conservation of norm

$$N=\int_{-\infty}^{\infty}|\psi|^2 d\tau, \tag{3}$$

momentum

$$P=i\int_{-\infty}^{\infty}(\psi_\tau^*\psi-\psi_\tau\psi^*)d\tau, \tag{4}$$

and energy

$$E=\frac{1}{2}\int_{-\infty}^{\infty}(|\psi_\tau|^2-|\psi|^4)d\tau. \tag{5}$$

For the system of two solitons having amplitudes a_1, a_2 and velocities V_1, V_2, the above conservation laws take the form

$$N=2\sum_{j=1}^{2}a_j, \quad P=4\sum_{j=1}^{2}a_jV_j,$$

$$E=\sum_{j=1}^{2}\left(a_jV_j^2-\frac{1}{3}a_j^3\right). \tag{6}$$

A weak perturbation reduces a completely integrable system to a nearly integrable one, and conservation laws (3)–(5) are, generally speaking, fulfilled only approximately. However we can claim that a *strong* exchange in norm, momentum, and energy between two colliding solitons is, in principle, possible in a *weakly* perturbed system because the exchange is not forbidden by the conservation laws that establish only three relations between four soliton parameters. In the following, we demonstrate numerically that such exchange does really happen, and it becomes very strong in a specific range of solitons' parameters even in the case of a weak perturbation.

The paper is organized as follows. In Sec. II, the exact two-soliton solution to unperturbed NLSE is analyzed, and the possibility of strong inelasticity of collision is predicted in a certain interval of relative phase of colliding solitons. In Sec. III we study numerically the influence of perturbations of two different types and confirm the predicted effects. Section IV concludes the paper.

II. TWO-SOLITON SOLUTION TO UNPERTURBED NLSE

The exact coherent two-soliton solution to the unperturbed NLSE can be found analytically [18], and we present this solution in the following form:

$$\psi(\xi,\tau)=\frac{1}{D}\Bigg\{\frac{a_1e^{i\gamma_1}}{\mathrm{ch}\beta_1}[a_1^2-a_2^2+(V_1-V_2)^2+2ia_2\mathrm{th}\,\beta_2 \times(V_1-V_2)]+\frac{a_2e^{i\gamma_2}}{\mathrm{ch}\,\beta_2}[a_2^2-a_1^2+(V_2-V_1)^2 +2ia_1\mathrm{th}\,\beta_1(V_2-V_1)]\Bigg\}, \tag{7}$$

where

$$D=(a_1-a_2)^2+(V_1-V_2)^2+2a_1a_2\left(\frac{\mathrm{ch}[\beta_1-\beta_2]-\cos[\gamma_1-\gamma_2]}{\mathrm{ch}\,\beta_1\mathrm{ch}\,\beta_2}\right), \tag{8}$$

$$\beta_j=a_j[\tau-\tau_j-V_j(\xi-\xi_j)],$$

$$\gamma_j=V_j(\tau-\tau_j)+(a_j^2-V_j^2)(\xi-\xi_j)/2,$$

and ξ_j,τ_j are the effective coordinate shifts ($j=1,2$). Hereafter in the paper, we use index $j=1,2$ when refer to both solitons. If at $\xi=0$ the solitons are at positions d_1 and d_2 with initial phases ϕ_1 and ϕ_2, then their effective coordinate shifts are found as

$$\xi_j=-2\frac{\mathrm{sgn}(d_{3-j}-d_j)(AV_j/a_j+\Phi_j)+\phi_j}{a_j^2+V_j^2},$$

$$\tau_j=\mathrm{sgn}(d_{3-j}-d_j)\frac{A}{a_j}+V_j\xi_j+d_j, \tag{9}$$

where

$$A=\frac{1}{2}\ln\left[\frac{(a_1+a_2)^2+(V_1-V_2)^2}{(a_1-a_2)^2+(V_1-V_2)^2}\right], \tag{10}$$

$$\Phi_j=\arg[a_j^2-a_{3-j}^2+(V_1-V_2)^2+i2a_{3-j}(V_j-V_{3-j})], \tag{11}$$

$\mathrm{sgn}(x)=-1,0,1$ for $x<0$, $x=0$, and $x>0$, respectively, and function arg is supposed to give values in the interval $[0,2\pi)$.

Thus a particular two-soliton solution is uniquely defined by eight parameters: soliton amplitudes a_j, velocities V_j, initial positions d_j, and initial phases ϕ_j. In our calculations we take $a_j>0$ because the change of the sign of soliton's amplitude is equivalent to initial phase shift by $\pm\pi$.

The angular frequency ω, period of oscillation T, and wavelength λ of a soliton can be expressed in terms of its amplitude and velocity as

$$\omega_j=\frac{1}{2}(a_j^2+V_j^2), \qquad T_j=\frac{2\pi}{\omega_j}, \qquad \lambda_j=T_j|V_j|. \tag{12}$$

A. Collision point

Well before the collision in unperturbed system the jth soliton moves on (ξ,τ) plane along one of the lines $\tau-\tau_c\pm A/a_j=V_j(\xi-\xi_c)$ and well after the collision along the other line. It is clear from the equations of the lines that the solitons experience the coordinate shift $\pm 2A/a_j$ due to the collision. The coordinates of the collision point are

$$\xi_c = \frac{1}{V_1 - V_2}\left[d_2 - d_1 - A\,\mathrm{sgn}(d_2 - d_1)\frac{a_1 + a_2}{a_1 a_2}\right],$$

$$\tau_c = \mathrm{sgn}(d_{3-j} - d_j)\frac{A}{a_j} + V_j \xi_c + d_j, \quad j = 1 \text{ or } 2. \tag{13}$$

The sign of ξ_c indicates that the system at $\xi = 0$ is either before $(+)$ or after $(-)$ or at the instant (0) of the collision.

Let us analyze the solution (7) at the collision point $\xi = \xi_c$, $\tau = \tau_c$. We have $\beta_j(\xi_c, \tau_c) = 0$ and

$$\gamma_j(\xi_c, \tau_c) = \omega_j \xi_c + \mathrm{sgn}(d_{3-j} - d_j)\left(\frac{AV_j}{a_j} + \Phi_j\right) + \phi_j. \tag{14}$$

It is not difficult to demonstrate that for given a_j, V_j, and d_j the squared absolute value of the solution $|\psi(\xi,\tau)|^2$ is a periodic function of the relative phase of the solitons $\Delta\gamma = \gamma_1 - \gamma_2$. At the collision point, it reaches its maximum possible value $|\psi(\xi_c, \tau_c)|^2 = (a_1 + a_2)^2$ when

$$\Delta\gamma_c = \gamma_1(\xi_c, \tau_c) - \gamma_2(\xi_c, \tau_c) = 2\pi m, \tag{15}$$

or, in terms of initial phases,

$$\Delta\phi = \phi_1 - \phi_2 = 2\pi m - \xi_c(\omega_1 - \omega_2) + \mathrm{sgn}(d_1 - d_2) \times \sum_{j=1}^{2}\left(\frac{AV_j}{a_j} + \Phi_j\right), \tag{16}$$

for an integer m.

For a particular case of symmetric collision, $a_1 = a_2$ and $V_1 = -V_2$, Eq. (16) reduces to

$$\Delta\phi = 2\pi m. \tag{17}$$

The condition of collision with maximum amplitude in the form of either Eq. (15) or Eq. (16) is important in the following discussion because the role of perturbation terms increases with increase in amplitude. Note that the maximum possible amplitude, $(a_1 + a_2)^2$, is proportional to squared norm of two-soliton solution [see Eq. (6)].

B. Comparison of two solutions with the same norm *N*, momentum *P*, and energy *E*

The aim of this section is to demonstrate that different two-soliton solutions with the same *N*, *P*, and *E* can be very close to each other at $\xi = \xi_c$ if the solitons' phases are properly chosen. This means that even a small perturbation would be enough for such solution to be transformed into another.

Let us consider two different two-soliton solutions of the type (7), $\psi(\xi,\tau)$ and $\tilde{\psi}(\xi,\tau)$. For the solution ψ, we fix all soliton parameters, a_j, V_j, d_j, and ϕ_2, except ϕ_1, which is considered as a free parameter.

For the solution $\tilde{\psi}$ we set magnitude for one of four parameters $\tilde{a}_j$, $\tilde{V}_j$, and find the other three parameters from Eq. (6) so that ψ and $\tilde{\psi}$ have the same *N*, *P*, and *E*. Initial positions of solitons $\tilde{d}_j$ are found with the use of Eq. (13) in a way that the coordinates of collision points for both solutions coincide, i.e., $\xi_c = \tilde{\xi}_c$ and $\tau_c = \tilde{\tau}_c$. Initial phases $\tilde{\phi}_j$ are determined from two conditions $\tilde{\gamma}_j(\xi_c, \tau_c) = \gamma_j(\xi_c, \tau_c)$.

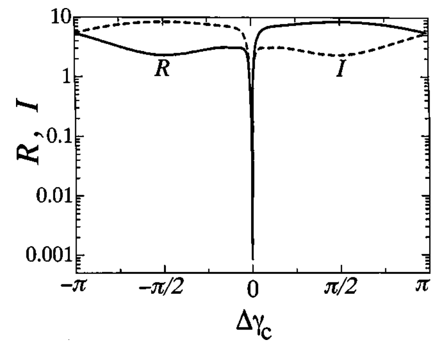

FIG. 1. Integrals $R(\Delta\gamma_c)$ and $I(\Delta\gamma_c)$ [Eqs. (18) and (19)] for two solutions $\psi(\xi,\tau)$ and $\tilde{\psi}(\xi,\tau)$ having the same *N*, *P*, and *E* ($a_1 = a_2 = 1$, $V_1 = -V_2 = 0.01$ and $\tilde{a}_1 = 1.1$, $\tilde{a}_2 = 0.9$, $\tilde{V}_1 \approx 0.0909$, $\tilde{V}_2 \approx -0.1111$). *R* and *I* drop by several orders of magnitude in a narrow vicinity of $\Delta\gamma_c = 0$.

We want to compare the solutions ψ and $\tilde{\psi}$ at the moment of collision $\xi = \xi_c$ for various magnitudes of relative phase $\Delta\gamma_c$. Note that, in our case, $\Delta\gamma_c$ is defined by the choice of $\Delta\phi_1$. As a measure of difference between $\psi(\xi_c,\tau)$ and $\tilde{\psi}(\xi_c,\tau)$ we use the following integrals:

$$R = \int_{-\infty}^{\infty} \{\mathrm{Re}[\psi(\xi_c,\tau)] - \mathrm{Re}[\tilde{\psi}(\xi_c,\tau)]\}^2 d\tau \tag{18}$$

and

$$I = \int_{-\infty}^{\infty} \{\mathrm{Im}[\psi(\xi_c,\tau)] - \mathrm{Im}[\tilde{\psi}(\xi_c,\tau)]\}^2 d\tau. \tag{19}$$

In Fig. 1, we plot $R(\Delta\gamma_c)$ and $I(\Delta\gamma_c)$ for two solutions with the same *N*, *P*, and *E*. As it is expected from Eq. (15), the dependences are 2π-periodic.

It is remarkable that both *R* and *I* drop by several orders of magnitude in a narrow vicinity of $\Delta\gamma_c = 0$. Figure 2 gives an insight of how close are the solutions $\psi(\xi_c,\tau)$ (solid circles) and $\tilde{\psi}(\xi_c,\tau)$ (open circles) at $\Delta\gamma_c = 0$. The curves, representing real parts of the solutions, are almost indistinguishable in the scale of the figure. The imaginary parts (not shown in Fig. 2) are similarly close. The results, presented in Figs. 1 and 2, were obtained for a typical set of parameters $a_1 = a_2 = 1$, $V_1 = -V_2 = 0.01$, and $\tilde{a}_1 = 1.1$, $\tilde{a}_2 = 0.9$, $\tilde{V}_1 \approx 0.0909$, $\tilde{V}_2 \approx -0.1111$.

We come to the conclusion that a pair of solitons colliding with relative phase $\Delta\gamma_c$, nearly satisfying Eq. (15), can be easily transformed by a weak perturbation into another pair with nearly same *N*, *P*, and *E*. The set of such pairs is infinite and, therefore, chaotic nature should be proper to mutual transformations of these pairs in a weakly perturbed system, when internal soliton modes are excluded from the consider-

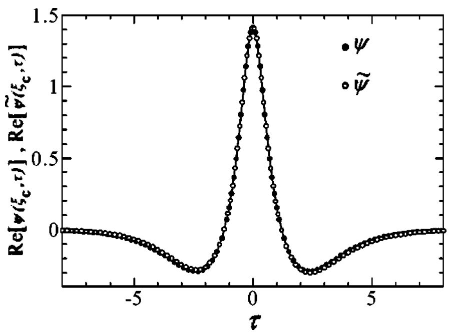

FIG. 2. Profiles of real parts of solutions $\psi(\xi_c,\tau)$ (solid circles) and $\tilde{\psi}(\xi_c,\tau)$ (open circles) at $\Delta\gamma_c=0$. Imaginary parts are similarly close. The soliton parameters are same as in Fig. 1.

ation. Since the conservation laws are almost fulfilled, the transformation can happen without significant radiation. Particular type of perturbation is not important because the above conclusions were made from the analysis of solutions to unperturbed NLSE.

III. NUMERICAL RESULTS

To integrate Eq. (1) numerically we use the implicit Crank-Nicolson method with the accuracy $O(\Delta\xi^2)$. Since we study the discreteness in τ, the influence of discreteness in ξ should be minimized. This can be achieved by setting $\Delta\xi=0.1\pi\Delta\tau^2$, where $\pi\Delta\tau^2$ is the shortest period of oscillations for the linearized discrete NLSE. The analytic two-soliton solution to unperturbed NLSE, Eq. (7), is taken as initial conditions. The reflecting boundary conditions are employed.

For perturbation parameters we assign values from the domains $\epsilon\in[-0.02,0.02]$ and $\Delta\tau\in[0.025,0.35]$. Both of them correspond to weak perturbation.

We integrate Eq. (1) until solitons go far apart after collision so that they can be treated as independent quasiparticles.

To study inelasticity of collision, with the use of Eqs. (3)–(5) we calculate the norms N_j', momenta P_j', and energies E_j' of solitons after collision and compare them with N_j, P_j, and E_j before the collision. Integration is made over the intervals of localization of each soliton. These intervals are centered at the soliton positions, found as the points, where $|\psi|^2$ has maxima. Sometimes, it is convenient to use the change in solitons' velocities V_j as a measure of inelasticity of the collision.

A collision is elastic when relative change in physical quantities is negligible for each soliton:

$$|\Delta X_j|=|X_j'-X_j|\ll|X_j|,\qquad X_j=N_j,P_j,E_j,V_j. \tag{20}$$

We call a collision inelastic if soliton parameters change significantly due to collision.

A. The role of phase difference. Chaotic nature of soliton collisions

In Sec. II, the possibility of radiationless transformation of one solution into another was predicted in a narrow range of solitons' relative phase $\Delta\gamma_c$. This may happen when $\Delta\gamma_c$ nearly satisfies condition (15). Here we verify this prediction numerically and find out other remarkable facts.

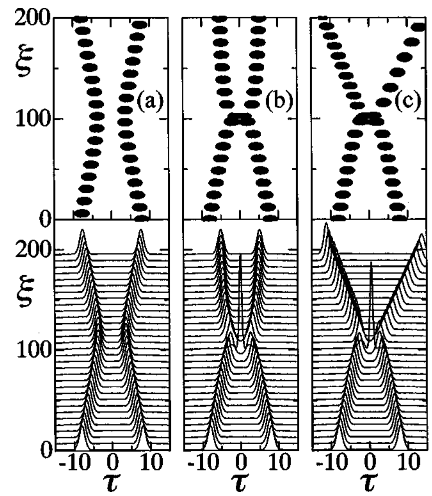

FIG. 3. Examples of soliton collisions for (a) $\Delta\gamma_c=\pi$, (b) $\Delta\gamma_c=0$, and (c) $\Delta\gamma_c=-0.08$. The collision in (a) is practically elastic, and collisions in (b) and (c) are strongly inelastic. Top images are the density plots for Re(ψ)$>$0.3. Bottom images are the pseudo-three-dimensional plots for $|\psi^2|$. Perturbation parameters are $\Delta\tau=0.025$, $\epsilon=-0.02$, and the solitons' parameters are $a_j=1$, $V_j=\pm0.05$, and $d_j=\mp8$.

Numerical results reveal only quantitative difference between collisions of symmetric and asymmetric solitons. Therefore, we restrict ourselves to the case of symmetric collision when the condition of inelastic collision in the form of Eq. (17) with the parameter $\Delta\phi$ can be used. Nevertheless, we prefer to use more general condition (15) with the parameter $\Delta\gamma_c$ to present results in the form applicable to both symmetric and asymmetric cases.

For each pair of solitons with certain a_j and V_j we fix initial positions d_j to make $\Delta\gamma_c$ be the only governing parameter. To vary $\Delta\gamma_c$ we put $\phi_2=0$ and vary only ϕ_1. As well as solution (7) is 2π periodic with respect to $\Delta\gamma_c$, it is sufficient to consider $\Delta\gamma_c\in[-\pi,\pi)$.

In Fig. 3 we show the pictures of soliton collision in (ξ,τ) plane for (a) $\Delta\gamma_c=\pi$, (b) $\Delta\gamma_c=0$, and (c) $\Delta\gamma_c=-0.08$. The top figures are density plots for Re(ψ)$>$0.3, and the bottom figures show $|\psi|^2$. The collision in (a) is practically elastic; in (b) and (c) the collisions are strongly inelastic. Figure 3 illustrates the fact that the antiphase solitons [in (a)] interact as mutually repulsive particles while the in-phase solitons [in (b) and (c)] interact as mutually attractive particles. Significant exchange in all three conserved quantities takes place in (c), while in (b) the change in amplitudes (norms) of solitons is small. As Eq. (13) predicts, the coordinates of the collision point (ξ_c,τ_c) do not depend on $\Delta\gamma_c$. For soliton parameters in Fig. 3 we set $a_j=1$, $V_j=\pm0.05$, $d_j=\mp8$, and perturbation parameters are $\Delta\tau=0.025$, and $\epsilon=-0.02$.

In Fig. 4 we plot velocities of the solitons $V_j'(\Delta\gamma_c)$ after collision. We show only the vicinity of $\Delta\gamma_c=0$, where, in line with Eq. (15), inelasticity of the collision drastically

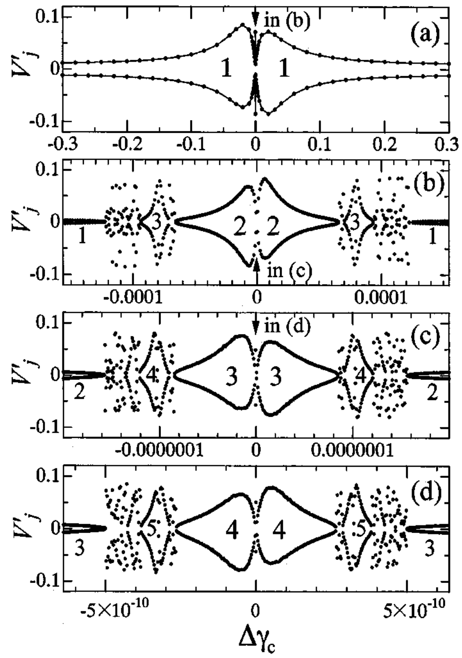

FIG. 4. Fractal scattering of two solitons, presented by velocities of solitons after collision V_j' as functions of relative phase $\Delta\gamma_c$. In (b)–(d) the subsequent blowups of the region near $\Delta\gamma_c=0$ are shown. Figures inside the smooth parts of the curves denote the number of collisions before solitons escape from each other. Soliton parameters are $a_j=1$, $V_j=\pm 0.01$, and $d_j=\mp 5$. Perturbation parameters are $\Delta\tau=0.2$ and $\epsilon=0$.

increases. One can see from Figs. 3 and 4(a) that inelasticity of the collision is very sensitive to the relative phase of solitons only in a narrow vicinity of $\Delta\gamma_c=0$; in the rest of the domain $[-\pi,\pi)$ the collision is practically elastic, as it is usually expected in weakly perturbed systems.

The curves in Fig. 4(a) demonstrate chaotic behavior as $|\Delta\gamma_c|$ becomes of the order of 10^{-4}. The blowup of this region, presented in (b), reveals the existence of intervals with apparently chaotic behavior of $V_j'(\Delta\gamma_c)$ alternating with the intervals of smooth behavior. Function $V_j'(\Delta\gamma_c)$ manifests the property of self-similarity, illustrated by the subsequent blowups in Figs. 4(b)–4(d). This self-similar pattern is related to the fractal soliton scattering.

The origin of fractal structure of function $V_j'(\Delta\gamma_c)$ has been explained in Ref. [15] for SGE and in Refs. [16,17] for NLSE. In the NLSE perturbed by the discreteness or by quintic term with $\epsilon<0$, solitons attract each other with a weak force. This type of attraction appears only in the presence of perturbation and it is different from the attraction between in-phase solitons, which exists also in the unperturbed system. The attraction, induced by perturbation, is responsible for the fractal soliton scattering. As a result of inelastic collision, solitons can gain a small relative velocity such that they cannot overcome mutual attraction. In this situation the solitons collide for the second time. In the second collision the solitons can acquire an amount of kinetic energy sufficient to escape each other. However, there exists a finite probability to gain the kinetic energy below the escape limit. While solitons keep colliding, they form a two-soliton bound state. The greater is the number of collisions before the escape, the longer is the lifetime of the bound state. Every smooth part of function $V_j'(\Delta\gamma_c)$ in Fig. 4 corresponds to a specific number of collisions before solitons escape each other. These numbers are marked in the figure for visible smooth parts in all blowups.

One can notice from Fig. 4 that there is the inverse symmetry of the curves with respect to the origin, and the inversion is opposite in odd and even blowups.

The fractal structure of Fig. 4 proves the chaotic nature of the soliton scattering. If the relative soliton phase $\Delta\gamma_c$ is a random variable, then the result of a particular collision cannot be predicted and it can be described only probabilistically. Sensitivity of the result of collision to relative phase $\Delta\gamma_c$ becomes extremely large when $\Delta\gamma_c$ nearly satisfies condition (15).

The lifetime of two-soliton bound states is discussed in detail in Ref. [17].

Here we would like to clarify the difference in nature of chaotic scattering in our case from the reported, e.g., in Refs. [11,12,14]. In any case the fractal scattering can be observed in the presence of only attractive perturbation, when two mutually attracted solitons can escape each other only if they have sufficiently large relative velocity. It has been reported that, with a finite probability, two solitons do not overcome mutual attraction even when they collide with a velocity greater than a threshold value because a part of their kinetic energy can be transferred into a soliton internal mode [11,12,14]. Energy of the internal modes can be transferred back into soliton kinetic energy during the second (or third etc.) collision and the two-soliton oscillatory system breaks up. The number of collisions before the breakup and the breakup velocity are very sensitive to the parameters of collision and thus the results of collisions have a fractal structure [11,12,14] as a function of collision parameters. This, however, occurs only if the perturbation is sufficiently large. Otherwise the soliton internal modes cannot be excited. On the other hand, too big perturbation kills the effect because of the fast radiation decay of the two-soliton oscillatory system.

In the present paper, the perturbation parameter is too small for soliton internal modes to be excited. Instead of excitation of internal soliton modes we demonstrated the possibility of radiationless energy transfer directly between kinetic and internal energy of the solitons (interchange in solitons' amplitudes and velocities). This energy exchange together with mutual attraction between solitons give the fractal scattering picture similar to that observed by other authors [11,12,14]. The physically important difference is that in our case the radiation losses are very small and lifetime of the oscillatory system can be very long. We observed breaking up of the oscillatory system into independent solitons after many hundreds of collisions, when the relative phase $\Delta\gamma_c$ was properly adjusted.

B. The role of discreteness $\Delta\tau$, quintic term ϵ, and collision velocity V

In order to study the influence of discreteness only, we put $\epsilon=0$. For a fixed $\Delta\tau$ we calculate $N_j'(\Delta\gamma_c)$, $P_j'(\Delta\gamma_c)$, and

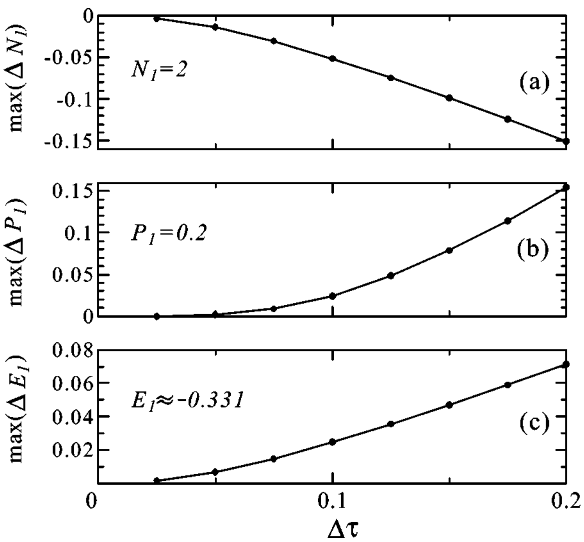

FIG. 5. Maximum possible changes of N_1, P_1, and E_1 due to collision as functions of $\Delta\tau$ at $\epsilon=0$, $a_j=1$, $V_j=\pm 0.05$, and $d_j=\mp 5$.

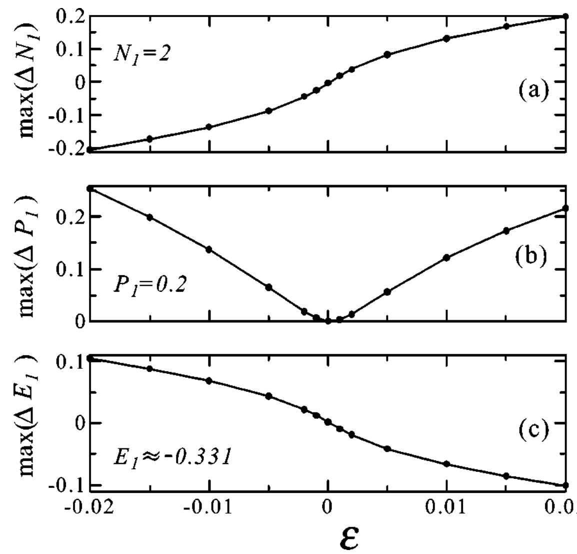

FIG. 6. Maximum possible changes of N_1, P_1, and E_1 due to collision as functions of ϵ at $\Delta\tau=0.025$, $a_j=1$, $V_j=\pm 0.05$, and $d_j=\mp 5$.

$E_j'(\Delta\gamma_c)$ after collision for $\Delta\gamma_c\in[-\pi,\pi)$ and compare them with the corresponding quantities before the collision. The results prove the suggestion that inelasticity of collision increases in the vicinity of $\Delta\gamma_c=0$. In Fig. 5, the maximum values of ΔN_1, ΔP_1, and ΔE_1 due to collision are shown as functions of $\Delta\tau$. One can see that weak discreteness indeed causes relatively strong inelasticity of collision. As an example, for $\Delta\tau=0.1$, the maximum changes in N_1, P_1, and E_1 are 2.5%, 11.8%, and 8.2%, respectively.

We do not plot changes in N, P, and E for the second soliton because they are practically the same as those for the first one taken with the opposite sign. The difference is in the third digit within the whole domain of $\Delta\gamma_c\in[-\pi,\pi)$. This fact suggests that the radiation is very small. The soliton parameters in Fig. 5 are equal to $V_j=\pm 0.05$, $a_j=1$, $d_j=\mp 5$.

In Fig. 6 we study the influence of the quintic term on maximum possible changes in N_1, P_1, and E_1. To minimize the influence of the discreteness, we set $\Delta\tau=0.025$, which causes (see Fig. 5 for $\epsilon=0$) the maximum changes in N_j, P_j, and E_j such as 0.17%, 0.05%, and 0.5%, respectively. The soliton parameters are the same as in Fig. 5. For the quintic perturbation, inelasticity of the collisions is strong as well. For $\epsilon=\pm 0.02$, the norm N_j (or amplitude) of a soliton, its momentum P_j, and energy E_j change due to collision by more than 10%, 100%, and 30%, respectively.

We emphasize again that such enormous sensitivity of the effect to $\Delta\tau$ and ϵ takes place only in a narrow interval of the soliton relative phase around the point $\Delta\gamma_c=0$. Outside of this interval in the domain $[-\pi,\pi)$, the influence of perturbations is several orders of magnitude weaker.

Another important parameter is the collision velocity that is $V_1-V_2=2V$ in the case of symmetric collision. The dependences, depicted in Fig. 7, manifest rapid increase in inelasticity of collision with decrease in collision velocity V. The inelasticity of collision is very strong for $V\ll 1$ and it is weak for $V\sim 1$. It is clear from the form of solution (7) and it was also verified numerically that inelasticity of collision becomes stronger in the case $|V_1-V_2|\rightarrow 0$, which is more general than the case $V_j\rightarrow 0$, studied numerically. Figure 7 was obtained for $\epsilon=-0.02$, $\Delta\tau=0.025$, $a_j=1$, and $d_j=\mp 5$.

Now let us consider influence of collision velocity and magnitude of perturbation parameter on the fractal scattering pattern. In Fig. 8(a) we plot the velocity after collision for the first soliton, V_1', as function of $(\Delta\gamma_c)$ for $\Delta\tau=0.3$ and different collision velocities V_j taken from the interval $V_1=-V_2\in[0.1,1.1]$ with the step 0.1. The width of the inter-

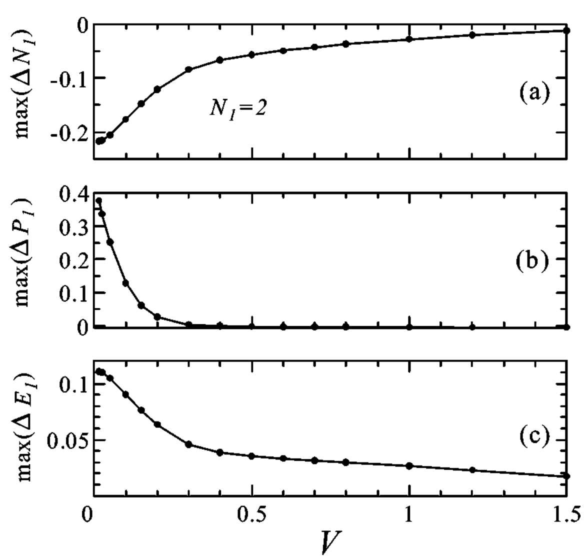

FIG. 7. Maximum possible changes of N_1, P_1, and E_1 due to collision as functions of $V=V_1=-V_2$ for $\epsilon=-0.02$, $\Delta\tau=0.025$, $a_j=1$, and $d_j=\mp 5$. The inelasticity of collision is very strong for $V\ll 1$ and it is weak for $V\sim 1$.

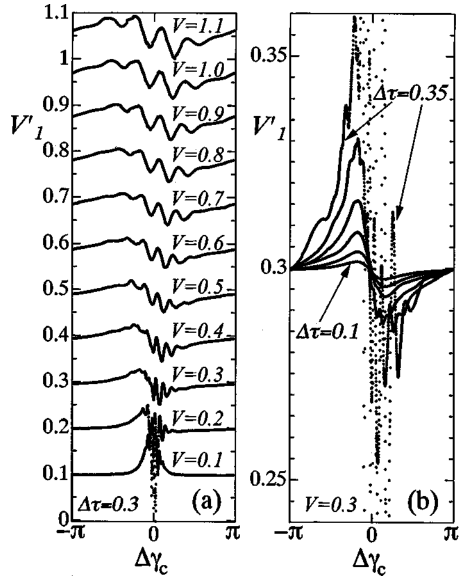

FIG. 8. Characteristic changes in chaotic scattering pattern due to variation of (a) collision velocity and (b) perturbation parameter. Variation ranges are $V_1=-V_2\in[0.1,1.1]$ with the step 0.1 for $\Delta\tau=0.3$ in (a) and $\Delta\tau\in[0.01,0.35]$ with the step 0.05 for $V_1=-V_2=0.3$ in (b). For both (a) and (b) we take $a_j=1$ and $\epsilon=0$. The width of the interval of inelastic collisions increases with increase in the collision velocity. The inelasticity of the collisions becomes stronger as the magnitude of perturbation parameter increases. The chaotic behavior cannot be observed for any $\Delta\gamma_c$ if the solitons' collision velocity is bigger than some limit, determined by the strength of the perturbation.

val of $(\Delta\gamma_c)$, where collisions are inelastic, increases on increase in collision velocity. The radiation grows as well however remains small. In Fig. 8(b), the function $V_1'(\Delta\gamma_c)$ is plotted for $V_1=-V_2=0.3$ and different discreteness parameter $\Delta\tau\in[0.01,0.35]$ with the step 0.05. Inelasticity of the collisions becomes stronger as magnitude of perturbation parameter increases.

It is interesting that the chaotic behavior of function $V_j'(\Delta\gamma_c)$ can be observed in Fig. 8(a) only for sufficiently small collision velocities and in Fig. 8(b) only for sufficiently strong perturbation. This means that, in a system with perturbation of a certain magnitude, the solitons can collide only once for any $\Delta\gamma_c$ if their collision velocity is bigger than some limit.

On the other hand, near the point $\Delta\gamma_c=0$, for solitons with small collision velocity, e.g., $V\sim10^{-3}$, the nonradiating collisions with significant *relative* exchange in physical quantities take place even for the discreteness parameter $\Delta\tau$ of the order of 10^{-2}. Therefore, the mentioned above calculation accuracy, for which the discreteness can be neglected, should be estimated separately for different collision velocities *V*.

One more remark is obvious from Fig. 4. Although the results of collision are extremely sensitive to the relative phase of the solitons, the height of the peaks of functions $V_j'(\Delta\gamma_c)$ remains the same in each blowup. This means that maximum intensity of the effect is determined only by the strength of the perturbation [Fig. 8(b)].

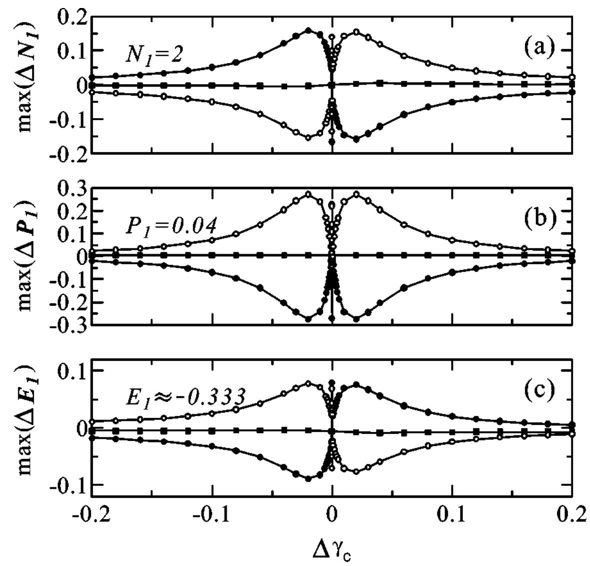

FIG. 9. Changes in (a) N_1, (b) P_1, and (c) E_1 due to collision as functions of relative phase of solitons $\Delta\gamma_c$ for different sets of perturbation parameters: $\epsilon=0.0105$, $\Delta\tau=0.025$ (open circles), $\Delta\tau=0.2$, $\epsilon=0$ (solid circles), and $\Delta\tau=0.2$, $\epsilon=0.0105$ (squares). Parameters of the solitons are $a_j=1$, $V_j=\pm0.01$, and $d_j=\mp5$.

C. Compensation effect

In Fig. 9 we plot changes in *N*, *P*, and *E* for one of the solitons due to the collision with another one as functions of solitons' relative phase $\Delta\gamma_c$. Three curves in each part of the figure correspond to different sets of perturbation parameters, while soliton parameters are the same $(a_j=1,\ V_j=\pm0.01,$ and $d_j=\mp5)$. Open circles show the case of $\epsilon=0.0105$ and $\Delta\tau=0.025$. It has been mentioned in Sec. III B that for this value of $\Delta\tau$ the discreteness effect can be neglected and the dominant role is played by the quintic perturbation term. On the other hand, solid circles show the influence of discreteness: $\Delta\tau=0.2$ and $\epsilon=0$. Finally, solid squares show the simultaneous action of both perturbations: $\Delta\tau=0.2$ and $\epsilon=0.01008$. One can see from Fig. 9 that the discreteness and the quintic term with $\epsilon>0$ make the effects of the opposite sign.

A fact of the exceptional importance is that, for properly chosen $\Delta\tau\neq0$ and $\epsilon\neq0$, these two perturbations compensate each other, and collisions become practically elastic for any relative phase parameter $\Delta\gamma_c$ (see the curve, marked with squares in Fig. 9).

This remarkable property can be important in applications where the strong inelasticity of soliton collisions is an unwanted consequence of an inevitable perturbation in the system. Effect of such a pertubation can be suppressed by perturbation of another type if the latter can be introduced and controlled artificially.

Another merit of the compensation effect is the possibility to establish correspondence between different kinds of perturbation. In Fig. 10 we show the relation between $\Delta\tau$ and ϵ when the influence of two perturbation terms in the right-hand side of Eq. (2) compensate each other and soliton collisions in the presence of both perturbations become almost elastic. Practically same result was obtained for symmetric

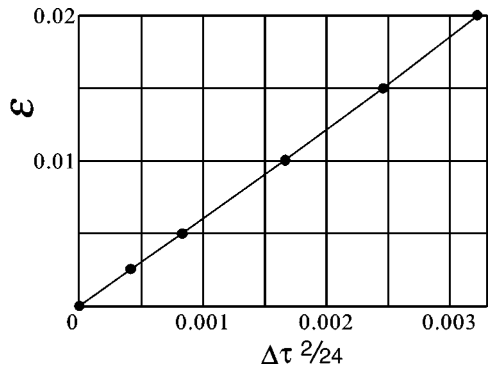

FIG. 10. Relation between $\Delta\tau$ and ϵ when the influence of two perturbation terms of the right-hand side of Eq. (2) compensate each other and soliton collisions in the presence of both perturbations become almost elastic. Practically the same result was obtained for symmetric ($a_j=1$, $V_j=\pm 0.05$, $d_j=\mp 5$) and asymmetric ($a_1=1.1$, $a_2=0.9$, $V_1=0.05$, $V_2=-0.1$, $d_j=\mp 5$) soliton pairs.

($a_j=1$, $V_j=\pm 0.05$, $d_j=\mp 5$) and asymmetric ($a_1=1.1$, $a_2=0.9$, $V_1=0.05$, $V_2=-0.1$, $d_j=\mp 5$) soliton pairs. The dependency $\epsilon(\Delta\tau^2/24)$ can be approximated by a straight line with the slope about 6.1 for each pair of solitons. With Fig. 10 in mind one can say that it is enough to study influence of only one weak perturbation and then the effect of perturbation of another kind can be predicted.

IV. DISCUSSION AND CONCLUSIONS

It was predicted and then confirmed numerically that, in a narrow range of the relative phase of solitons the collisions are strongly inelastic even for a very weak perturbation, when no internal soliton modes can be excited. A comparatively large exchange in norm, momentum, and energy occurs with practically no radiation so that conservation laws (3)–(5) are fulfilled with a high accuracy for the two-soliton system.

Since the effect was predicted in Sec. II from the analysis of the unperturbed NLSE, one can deduce that actual type of perturbation is not really important. Moreover, in Sec. III C, we established a relation between two different perturbation terms in the sense that they cause the same N, P, and E exchange during inelastic collision. It is likely that the total effect of several weak perturbations can be presented as a superposition of the contributions from each sort of perturbation. In particular, the effects of two different perturbations can compensate each other. The compensation mechanism may be used in applications to suppress an inevitable perturbation by introducing a controlled perturbation of another type in the system.

Our results can significantly change the understanding of soliton gas model. In the presence of even weak perturbation, the collisions of solitons can be inelastic. This is especially important in the case when solitons have small relative velocities because, in this case, the probability of inelastic collision increases (see Sec. III B).

The fractal scattering pattern proves the chaotic character of soliton collisions in weakly perturbed NLSE.

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